THE GROWTH OF GRIGORCHUK'S TORSION GROUP

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Abstract. In 1980 Rostislav Grigorchuk constructed a group G of intermediate growth, and later obtained the following estimates on its growth γ [Gri84]:

$$e^{\sqrt{n}} \precsim \gamma(n) \precsim e^{n^\beta},$$

where $\beta = \log_{32}(31) \approx 0.991$. Using elementary methods we improve the upper bound to

$$\gamma(n) \precsim e^{n^\alpha},$$

where $\eta \approx 0.811$ is the real root of the polynomial $X^3 + X^2 + X - 2$ and $\alpha = \log(2)/\log(2/\eta) \approx 0.767$.

1. Introduction

The notion of growth for finitely generated groups was introduced in the 1950's in the former Ussr [Sva55] and in the 1960's in the West [Mil68]. There are well-known classes of groups of polynomial growth (abelian, and more generally virtually nilpotent groups [Gro81]) and of exponential growth (non-virtually-nilpotent linear [Tit72] or non-elementary hyperbolic [GH90] groups). However, the first example of a group of intermediate growth was discovered later, by Rostislav Grigorchuk; see [Gri83, Gri84, Gri91]. He showed that the growth γ of his group satisfies

$$e^{\sqrt{n}} \precsim \gamma(n) \precsim e^{n^\beta},$$

where $\beta = \log_{32}(31) \approx 0.991$; see below for the precise definition of growth. The purpose of this note is to prove the following improvement:

Theorem 1.1. *Let η be the real root of the polynomial $X^3 + X^2 + X - 2$, and set $\alpha = \log(2)/\log(2/\eta) \approx 0.767$. Then the growth γ of Grigorchuk's group satisfies*

$$e^{\sqrt{n}} \precsim \gamma(n) \precsim e^{n^\alpha}.$$

2. Growth of Groups

Let G be a group generated as a monoid by a finite set S. A *weight* on (G, S) is a function $\omega : S \to \mathbb{R}_+^*$. It induces a *length* ∂_ω on G by

$$\partial_\omega : \begin{cases} G \to \mathbb{R}_+ \\ g \mapsto \min\{\omega(s_1) + \cdots + \omega(s_n) |\, s_1 \cdots s_n =_G g\}. \end{cases}$$

A *minimal form* of $g \in G$ is a representation of g as a word of minimal length over S. The *growth* of G with respect to ω is then

$$\gamma_\omega : \begin{cases} \mathbb{R}_+ \to \mathbb{R}_+ \\ n \mapsto \#\{g \in G |\, \partial_\omega(g) \le n\}. \end{cases}$$

The function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is *dominated* by $\delta : \mathbb{R}_+ \to \mathbb{R}_+$, written $\gamma \precsim \delta$, if there is a constant $C \in \mathbb{R}_+$ such that $\gamma(n) \le \delta(Cn)$ for all $n \in \mathbb{R}_+$. Two functions $\gamma, \delta : \mathbb{R}_+ \to \mathbb{R}_+$ are *equivalent*, written $\gamma \sim \delta$, if $\gamma \precsim \delta$ and $\delta \precsim \gamma$.

The following lemmata are well known:

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Lemma 2.1. *Let S and S' be two finite generating sets for the group G, and let ω and ω' be weights on (G,S) and (G,S') respectively. Then $\gamma_\omega \sim \gamma_{\omega'}$.*

Proof. Let $C = \max_{s\in S} \partial_{\omega'}(s)/\omega(s)$. Then $\partial_{\omega'}(g) \leq C\partial_\omega(g)$ for all $g \in G$, and thus $\gamma_\omega(n) \leq \gamma_{\omega'}(Cn)$, from which $\gamma_\omega \precsim \gamma_{\omega'}$. The opposite relation holds by symmetry. □

The *growth type* of a finitely generated group G is the $\sim$-equivalence class containing its growth functions; it will be denoted by γ_G.

Note that all exponential functions b^n are equivalent, and polynomial functions of different degree are inequivalent; the same holds for the subexponential functions e^{n^α}. We have

$$0 \precnsim n \precnsim n^2 \cdots \precnsim e^{\sqrt{n}} \precnsim \cdots \precnsim e^n.$$

Note also that the ordering $\precsim$ is not linear.

Lemma 2.2. *Let G be a finitely generated group. Then $\gamma_G \precsim e^n$.*

Proof. Choose for G a finite generating set S, and define the weight ω by $\omega(s) = 1$ for all $s \in S$. Then $\gamma_\omega(n) \leq |S|^n$ for all n, so $\gamma_G \precsim e^n$. □

If there is a $d \in \mathbb{N}$ such that $\gamma_G \precsim n^d$, the group G is of *polynomial growth* of degree at most d; if $\gamma_G \sim e^n$, then G is of *exponential growth*; otherwise G is of *intermediate growth.* The existence of groups of intermediate growth was first shown by Grigorchuk [Gri83].

3. The Grigorchuk 2-group

Let Σ^* be the set of finite sequences over $\Sigma = \{0,1\}$. For $x \in \Sigma$ set $\overline{x} = 1-x$. Define recursively the following length-preserving permutations of Σ^*:

$$\begin{gathered} a(x\sigma) = \overline{x}\sigma; \\ \begin{aligned} b(0\sigma) &= 0a(\sigma), & b(1\sigma) &= 1c(\sigma); \\ c(0\sigma) &= 0a(\sigma), & c(1\sigma) &= 1d(\sigma); \\ d(0\sigma) &= 0\sigma, & d(1\sigma) &= 1b(\sigma). \end{aligned} \end{gathered}$$

Then G, the Grigorchuk 2-group [Gri80, Gri84], is the group generated by $S = \{a,b,c,d\}$. It is readily checked that these generators are of order 2 and that $V = \{1,b,c,d\}$ is a Klein group.

Let $H = V^G$ be the normal closure of V in G. It is of index 2 in G and preserves the first letter of sequences; i.e. $H\cdot x\Sigma^* \subset x\Sigma^*$ for all $x \in \Sigma$. There is a map $\psi : H \to G\times G$, written $g \mapsto (g_0, g_1)$, defined by $0g_0(\sigma) = g(0\sigma)$ and $1g_1(\sigma) = g(1\sigma)$. As $H = \langle b,c,d,b^a,c^a,d^a\rangle$, we can write ψ explicitly as

$$\psi : \begin{cases} b \mapsto (a,c), & b^a \mapsto (c,a) \\ c \mapsto (a,d), & c^a \mapsto (d,a) \\ d \mapsto (1,b), & d^a \mapsto (b,1). \end{cases}$$

4. The Growth of G

Let $\eta \approx 0.811$ be the real root of the polynomial $X^3 + X^2 + X - 2$, and define the following function on S:

$$\begin{aligned} \omega(a) &= 1-\eta^3 = \eta^2+\eta-1, & \omega(c) &= 1-\eta^2, \\ \omega(b) &= \eta^3 = 2-\eta-\eta^2, & \omega(d) &= 1-\eta. \end{aligned}$$

It is a weight, because it takes positive values on every generator.

Lemma 4.1. *Every $g \in G$ admits a minimal form*

$$[*]a*a*a\cdots*a[*],$$

where $* \in \{b,c,d\}$ and the first and last $*$s are optional.

Proof. Clearly $\omega(s) > 0$ for $s \in S$, so ω is a weight. Let w be a minimal form of g. The lemma asserts that one can suppose there are no consecutive letters in $\{b, c, d\}$ in w; now two equal consecutive letters cancel, and the product of any two distinct letters in $\{b, c, d\}$ equals the third one. For any arrangement (x, y, z) of $\{b, c, d\}$ we have $\omega(x) \le \omega(y) + \omega(z)$, so the substitution of z for xy will not increase the weight of w. □

Proposition 4.2. *Let $g \in H$, with $\psi(g) = (g_0, g_1)$. Then*

$$\eta\big(\partial_\omega(g) + \omega(a)\big) \ge \partial_\omega(g_0) + \partial_\omega(g_1).$$

Proof. Let w be a minimal form of g. Thanks to Lemma 4.1 we may suppose the number of $*$s in w is at most the number of as plus one. Construct words w_0, w_1 over S using ψ seen as a substitution on words; they represent g_0 and g_1 respectively. Note that

$$\begin{aligned}\eta(\omega(a) + \omega(b)) &= \omega(a) + \omega(c),\\ \eta(\omega(a) + \omega(c)) &= \omega(a) + \omega(d),\\ \eta(\omega(a) + \omega(d)) &= 0 + \omega(b).\end{aligned}$$

As $\psi(b) = (a, c)$ and $\phi(aba) = (c, a)$, each b in w contributes $\omega(a) + \omega(c)$ to the total weight of w_0 and w_1; the same argument applies to c and d. Now, grouping together pairs of $*$s in $\{b, c, d\}$ and as, we see that $\eta(\partial_\omega(g))$ is a sum of left-hand terms, possibly $-\eta\omega(a)$; while $\partial_\omega(g_0) + \partial_\omega(g_1)$ is bounded by the total weight of the letters in w_0 and w_1, which is precisely the sum of the corresponding right-hand terms. □

Let $\alpha = \log(2)/\log(2/\eta) \approx 0.767$, and for $n \in \mathbb{N}$ set $C_n = \frac{1}{n+1}\binom{2n}{n}$, the nth Catalan number; remember that it is the number of labelled binary rooted trees with $n + 1$ leaves [Cat38, LW92, page 119].

Proposition 4.3. *Let $\zeta = \frac{\omega(a)}{2/\eta - 1}$, let $K > \zeta$ be any constant, and for $n \in \mathbb{R}_+$ set*

$$L_n = \max\left\{1, \left\lceil 2\left(\frac{n - \zeta}{K - \zeta}\right)^{\alpha}\right\rceil - 1\right\}.$$

Then we have

$$\gamma_\omega(n) \le C_{L_n - 1} 2^{L_n - 1} \gamma_\omega(K)^{L_n}. \tag{1}$$

Proof. We construct an injection ι of G into the set of labelled binary rooted trees each of whose leaves is labelled by an element of G of weight bounded by K and each of whose interior vertices is labelled by an element of the subgroup $\langle a \rangle$ of G. For $g \in G$, $\iota(g)$ is called its *representation*. It is constructed as follows: if $g \in G$ satisfies $\partial_\omega(g) \le K$, its representation is a tree with one vertex labelled by g. If $\partial_\omega(g) > K$, let $h \in \langle a \rangle$ be such that $gh \in H$, and write $\psi(gh) = (g_0, g_1)$. By Proposition 4.2, $\partial_\omega(g_i) \le \eta \partial_\omega(g)$, so we may construct inductively the representations of g_0 and g_1. The representation of g is a tree with h at its root vertex and $\iota(g_0)$ and $\iota(g_1)$ attached to its two branches.

We first claim that ι is injective: let $\mathcal{T}$ be a tree in the image of ι. If $\mathcal{T}$ has one node labelled by g, then $\iota^{-1}(\mathcal{T}) = \{g\}$. If $\mathcal{T}$ has more than one vertex, let $h \in \langle a \rangle$ be the label of the root vertex and $(\mathcal{T}_0, \mathcal{T}_1)$ be the two subtrees connected to the root vertex. By induction on the number of vertices of $\mathcal{T}$, we have $\mathcal{T}_i = \iota(g_i)$ for unique g_0 and g_1. Then as ψ is injective there is a unique $g \in G$ with $\psi(gh) = (g_0, g_1)$, and $\iota^{-1}(\mathcal{T}) = \{g\}$.

We next prove by induction on n that if $\partial_\omega(g) \le n$ then its representation is a tree with at most L_n leaves. Indeed if $n \le K$ then g's representation has one leaf and $L_n = 1$, while otherwise g's representation is made up of those of g_0 and g_1. Say $\partial_\omega(g_0) = \ell$ and $\partial_\omega(g_1) = m$; then by Proposition 4.2 we have $\ell + m \le \eta(n + \omega(a))$. By induction these representations have at most L_ℓ and L_m leaves. As $\alpha < 1$, we have $L_\ell + L_m \le 2L_{(\ell+m)/2}$ for all ℓ, m; and by direct computation, $L_{\eta/2(n+\omega(a))} = \lfloor L_n/2 \rfloor$, so the number of leaves of g's representation is

$$L_\ell + L_m \le 2L_{(\ell+m)/2} \le 2L_{\eta/2(n+\omega(a))} \le L_n,$$

as was claimed.

We conclude that $\gamma(n)$ is bounded by the number of representations with L_n leaves; there are C_{L_n-1} binary trees with L_n leaves, 2 choices of labelling for each of the $L_n - 1$ interior vertices, and $\gamma(K)$ choices for each leaf; so Equation (1) follows. □

A lower bound on the growth of G comes from the fact that G is residually a 2-group:

Theorem 4.4 (Grigorchuk [Gri89]). *Suppose G is a finitely generated residually-p group. Let $\{G_n\}$ be the Zassenhaus filtration of G. If $[G : G_n] \gnsim n^d$ for all d, then $\gamma_G \gtrsim e^{\sqrt{n}}$.*

Proof of Theorem 1.1. The sequence $[G : G_n]$ was shown to be of superpolynomial growth in [Gri89], so Theorem 4.4 yields the claimed lower bound; an elementary proof of this lower bound appears also in [Gri84].

For the upper bound, which is the main result of the present note, we invoke Proposition 4.3 with $K = 1$, noting that $L_n \sim n^\alpha$ and $C_{L_n} \leq 4^{L_n}$, to obtain $\gamma_\omega \precsim (4 \cdot 2 \cdot \gamma(1))^{n^\alpha} \sim e^{n^\alpha}$. □

5. Conclusion

The main fact used in the proof of Theorem 1.1 is the existence of minimal forms given by Lemma 4.1, coming from the natural map $\langle a \rangle * V \twoheadrightarrow G$. One can impose stronger conditions on minimal forms, such as 'not containing *dada* as a subword', coming from an explicit recursive presentation of G [Lys85]. Tighter upper bounds result from such considerations. Yurij Leonov [Leo98] recently obtained improvements on the lower bound of Theorem 1.1.

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